POLYNOMIAL ESTIMATES AND RADIUS OF ANALYTICITY ON REAL BANACH SPACES

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Abstract. A generalization of Problem 73 of Mazur and Orlicz in the Scottish Book was introduced from L. Harris. The exact value of the constant that appears there is known when complex normed linear spaces are considered. In this paper, we give estimates in the case of an arbitrary real normed linear space and a real ℓ_p space. Moreover, if $F(x)$ is a power series, ρ its radius of uniform convergence and ρ_A its radius of analyticity, we prove that $\rho_A \geq \rho/\sqrt{2}$ and give some respective results for the nth Fréchet derivative of $F(x)$.

1. introduction

If X is a Banach space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we let $\mathcal{L}^s({}^m X)$ denote the Banach space of all continuous symmetric m-linear forms $L : X^m \to \mathbb{K}$ with the norm

$$\|L\| = \sup\{|L(x_1, \dots, x_m)| : \|x_1\| \leq 1, \dots, \|x_m\| \leq 1\}.$$

A function $P : X \to \mathbb{K}$ is a continuous m-homogeneous polynomial if there is a continuous symmetric m-linear form $L : X^m \to \mathbb{K}$ for which $P(x) = L(x, \dots, x)$ for all $x \in X$. In this case it is convenient to write $P = \widehat{L}$. Let $\mathcal{P}({}^m X)$ denote the Banach space of all continuous m-homogeneous polynomials $P : X \to \mathbb{K}$ with the norm

$$\|P\| = \sup\{|P(x)| : \|x\| \leq 1\}.$$

We write $L(x_1^{k_1} \dots x_n^{k_n})$ as shorthand for $L(x_1, \dots, x_1, \dots, x_n, \dots, x_n)$ where each x_i appears k_i times for $1 \leq i \leq n$, $k_1 + \ldots + k_n = m$.
We also need to define the norm

$$\|L\|_{(n)} = \sup_{k_1 + \ldots k_n = m} \sup\{|L(x_1^{k_1} \dots x_n^{k_n})| : \|x_1\| \leq 1, \dots, \|x_n\| \leq 1\},$$

which for $n = 2$ gives

$$\|L\|_{(2)} = \sup_{1 \leq k \leq m} \sup\{|L(x_1^k x_2^{m-k})| : \|x_1\| \leq 1, \|x_2\| \leq 1\}.$$

Clearly, $\|\widehat{L}\| \leq \|L\|_{(n)} \leq \|L\|_{(n+1)} \leq \|L\|$.

It is known that if $L \in \mathcal{L}^s({}^m X)$ and $\widehat{L}$ the associated polynomial, then

$$\|L\| \leq \frac{m^m}{m!} \|\widehat{L}\|.$$

This is the answer of Problem 73 of Mazur and Orlicz in [11].

A natural generalization of Problem 73 is the following:
Let X be a normed linear space, $k_1,\dots,k_n$ be nonnegative integers whose sum is m and let $c(k_1,\dots,k_n,X)$ be the smallest number with the property that if L is any symmetric m-linear mapping of one real normed linear space into another, then

$$|L(x_1^{k_1}\dots x_n^{k_n})| \le c(k_1,\dots,k_n,X)\|\widehat{L}\|.$$

It is shown in [4] Theorem 1, that if only complex normed linear spaces and comlex scalars are considered, then

$$c(k_1,\dots,k_n,X) = \frac{k_1!\dots k_n!}{k_1^{k_1}\dots k_n^{k_n}}\frac{m^m}{m!}.$$

In the next two sections we shall find bounds for the constant $c(k_1,\dots,k_n,X)$ in the case of real normed linear spaces and specifically in the case of real ℓ_p spaces.

2. Polynomials on a real normed linear space

From [5] Corollary 7 (see also [8], [10]), we have

$$|L(x_1^{k_1}\dots x_n^{k_n})| \le \sqrt{\frac{m^m}{k_1^{k_1}\dots k_n^{k_n}}}\|\widehat{L}\| \tag{1}$$

for all non-negative integers $k_1,\dots,k_n$ with $k_1+\dots+k_n=m$.
For $n=2$ we get

$$|L(x_1^k x_2^{m-k})| \le \sqrt{\frac{m^m}{k^k(m-k)^{m-k}}}\|\widehat{L}\|$$

and we can easily see that the square root takes its maximum value which is $(\sqrt{2})^m$ when $k=\frac{m}{2}$.
To see this, simply consider the function $f(k)=k^k(m-k)^{m-k}$.
Then,

$$\begin{aligned} f'(k) &= (1+\ln k)k^k(m-k)^{m-k} - k^k[1+\ln(m-k)](m-k)^{m-k} \\ &= k^k(m-k)^{m-k}\ln\frac{k}{m-k} \end{aligned}$$

and

$$f'(k)=0 \Leftrightarrow \ln\frac{k}{m-k}=0 \Leftrightarrow \frac{k}{m-k}=1 \Leftrightarrow k=\frac{m}{2},$$

since $k\neq 0$ and $k\neq m$.
Moreover, $f'(k)<0$ for $0<k<\frac{m}{2}$ (i.e. f if strictly decreasing for $0<k<\frac{m}{2}$) and $f'(k)>0$ for $\frac{m}{2}<k<m$ (i.e. f if strictly increasing for $\frac{m}{2}<k<m$).
Therefore, the minimum value of f is $f\left(\frac{m}{2}\right)=\left(\frac{m}{2}\right)^m$. Thus,

$$\left(\frac{\|L\|_{(2)}}{\|\widehat{L}\|}\right)^{\frac{1}{m}} \le \sqrt{2}.$$

If m is odd, the previous inequality is strict (since $k \in \mathbb{N}$).
Similarly, we get the following general result:

$$\left(\frac{\|L\|_{(n)}}{\|\widehat{L}\|}\right)^{\frac{1}{m}} \leq \sqrt{n}$$

From [7] Lemma 3, we have

$$|L(x_1^{k_1}\dots x_n^{k_n})| \leq e^{\frac{m}{2}} C_m \binom{m+n-1}{n-1} \|\widehat{L}\|.$$

The value of the constant C_m needs to be determined in order to compare this result with (1). To do this, we need to find a bound for $\mathbb{E}(|A|^k)$ which is used in the proof of [7] Lemma 3. Stirling's formula and some elementary calculations will do the job.
For $k \neq 2$ even:

$$\begin{aligned}
\mathbb{E}(|A|^k) &\leq k(2p)^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right) = k(2p)^{\frac{k}{2}}\left(\frac{k}{2}-1\right)! = k(2p)^{\frac{k}{2}}\left(\frac{\frac{k}{2}-1}{e}\right)^{\frac{k}{2}-1}\sqrt{\frac{k}{2}-1} \\
&= k(2p)^{\frac{k}{2}}\left(\frac{k-2}{2e}\right)^{\frac{k}{2}}\frac{2e}{k-2}\sqrt{\frac{k-2}{2}} = \frac{ke\sqrt{2}}{\sqrt{k-2}}\left[\frac{p(k-2)}{e}\right]^{\frac{k}{2}} \\
&= \frac{ke\sqrt{2}}{\sqrt{k-2}}\left(\frac{k-2}{k}\right)^{\frac{k}{2}}\left(\frac{pk}{e}\right)^{\frac{k}{2}} = e\sqrt{2}\sqrt{k-2}\left(\frac{k-2}{k}\right)^{\frac{k}{2}-1}\left(\frac{pk}{e}\right)^{\frac{k}{2}} \\
&\leq e\sqrt{2(k-2)}\left(\frac{pk}{e}\right)^{\frac{k}{2}}.
\end{aligned}$$

For $k \neq 1$ odd:

$$\begin{aligned}
\mathbb{E}(|A|^k) &\leq k(2p)^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right) \leq k(2p)^{\frac{k}{2}}\Gamma\left(\frac{k+1}{2}\right) = k(2p)^{\frac{k}{2}}\left(\frac{k-1}{2}\right)! \\
&= k(2p)^{\frac{k}{2}}\left(\frac{\frac{k-1}{2}}{e}\right)^{\frac{k-1}{2}}\sqrt{\frac{k-1}{2}} = k(2p)^{\frac{k}{2}}\left(\frac{k-1}{2e}\right)^{\frac{k}{2}}\sqrt{\frac{2e}{k-1}}\sqrt{\frac{k-1}{2}} \\
&= k\sqrt{e}\left[\frac{p(k-1)}{e}\right]^{\frac{k}{2}} = k\sqrt{e}\left(\frac{k-1}{k}\right)^{\frac{k}{2}}\left(\frac{pk}{e}\right)^{\frac{k}{2}} \\
&= (k-1)\sqrt{e}\left(\frac{k-1}{k}\right)^{\frac{k}{2}-1}\left(\frac{pk}{e}\right)^{\frac{k}{2}} \leq (k-1)\sqrt{e}\left(\frac{pk}{e}\right)^{\frac{k}{2}}.
\end{aligned}$$

Moreover, $\mathbb{E}(|A|) \leq \sqrt{2p}$ and $\mathbb{E}(|A|^2) \leq 4p$.
In order to make the next calculations easier, we have to unite all the above cases. Thus, we take $\mathbb{E}(|A|^k) \leq ke\left(\frac{pk}{e}\right)^{\frac{k}{2}}$. The term ke could be slightly better, but it would not make any essential difference.
At the end of the proof of [7] Lemma 3, we need to calculate the $\sup_{k_i}(C_{k_1}\dots C_{k_n})$,

where $k_1 + \ldots + k_n = m$. We have:

$$\sup_{k_i}(C_{k_1} \ldots C_{k_n}) = \sup_{k_i}(k_1 e \ldots k_n e) = e^n \sup_{k_i}(k_1 \ldots k_n) \leq e^n \left(\frac{m}{n}\right)^n = \left(\frac{em}{n}\right)^n$$

Finally, we get

$$|L(x_1^{k_1} \ldots x_n^{k_n})| \leq e^{\frac{m}{2}} \left(\frac{em}{n}\right)^n \binom{m+n-1}{n-1} \|\widehat{L}\|. \tag{2}$$

Inequality (2) is worse than inequality (1). But asymptotically it gives better estimate, so it can be useful in some cases.

Combining inequalities (1) and (2) we obtain the following

Lemma 2.1. Let $L : X^m \to Y$ be an m-linear map. Then

$$\left(\frac{\|L\|_{(n)}}{\|\widehat{L}\|}\right)^{\frac{1}{m}} \leq \begin{cases} \sqrt{2}, \text{ if } n = 2 \\ C\sqrt{e}, \text{ if } n \geq 3 \end{cases}$$

where $C = C(m, n)$ is independent of L, X, Y and tends to 1 as $m \to \infty$ for fixed n.

The nth Rademacher function is defined on $[0, 1]$ by $r_n = \operatorname{sign} \sin 2^n \pi t$.

Lemma 2.2. (**Polarization Formula**) Let X be a vector space and $L \in \mathcal{L}^s({}^m X)$. If $x_1, \ldots, x_m \in X$, then

$$L(x_1, \ldots, x_m) = \frac{1}{m!} \int_0^1 r_1(t) \ldots r_m(t) \widehat{L}\left[\sum_{i=1}^m r_i(t) x_i\right] dt. \tag{3}$$

The proof of Lemma 2.2 is easy, so it is omitted (see [9] Lemma 2).

Proposition 2.3. Let X, Y be real normed linear spaces and $L : X^m \to Y$ be a continuous symmetric m-linear mapping with associated homogeneous polynomial $\widehat{L}$. If $x_1, \ldots, x_n$ are norm-one vectors in X, then

$$\frac{|L(x_1^{k_1} \ldots x_n^{k_n})|}{\|\widehat{L}\|} \leq \min\left\{\sqrt{\frac{m^m}{k_1^{k_1} \ldots k_n^{k_n}}}, \frac{k_1^{k_1} \ldots k_n^{k_n}}{m!} n^m\right\}$$

for all non-negative integers $k_1, \ldots, k_n$ with $k_1 + \ldots + k_n = m$.

Proof. Using Lemma 2.2, we have

$$L(x_1^{k_1} \ldots x_n^{k_n}) = \frac{1}{m!} \int_0^1 r_1(t) \ldots r_m(t) \widehat{L}[(r_1(t) + \ldots + r_{k_1}(t)) x_1 + \ldots] dt$$

$$\Leftrightarrow L\left(\left(\frac{x_1}{k_1}\right)^{k_1} \ldots \left(\frac{x_n}{k_n}\right)^{k_n}\right) = \frac{1}{m!} \int_0^1 r_1(t) \ldots r_m(t) \widehat{L}\left[(r_1(t) + \ldots + r_{k_1}(t)) \frac{x_1}{k_1} + \ldots\right] dt$$

$$\Leftrightarrow \frac{1}{k_1^{k_1} \ldots k_n^{k_n}} L(x_1^{k_1} \ldots x_n^{k_n}) = \frac{1}{m!} \int_0^1 r_1(t) \ldots r_m(t) \widehat{L}\left[(r_1(t) + \ldots + r_{k_1}(t)) \frac{x_1}{k_1} + \ldots\right] dt$$

Therefore,
(4)
$$
\begin{aligned}
|L(x_1^{k_1}\dots x_n^{k_n})| &\le \frac{k_1^{k_1}\dots k_n^{k_n}}{m!}\|\widehat{L}\|\int_0^1 \left\|(r_1(t)+\dots+r_{k_1}(t))\frac{x_1}{k_1}+\dots\right\|^m dt\\
&\le \frac{k_1^{k_1}\dots k_n^{k_n}}{m!}\|\widehat{L}\|\int_0^1 \left(\frac{|r_1(t)+\dots+r_{k_1}(t)|}{k_1}+\dots\right)^m dt\\
&\le \frac{k_1^{k_1}\dots k_n^{k_n}}{m!}n^m\|\widehat{L}\|.
\end{aligned}
$$
Inequalities (1) and (4) complete the proof. ☐

Asymptotically, inequality (4) gives much worse estimate than inequalities (1) and (2). It is useful though for large n's and when n depends on m, while inequality (1) is useful only for fixed n's. We now conclude that:

Theorem 2.4. Let X, Y be real normed linear spaces and $L : X^m \to Y$ be a continuous symmetric m-linear mapping with associated homogeneous polynomial $\widehat{L}$. If $x_1,\dots,x_n$ are norm-one vectors in X, then
$$
\frac{k_1!\dots k_n!}{k_1^{k_1}\dots k_n^{k_n}}\frac{m^m}{m!} \le c(k_1,\dots,k_n,X) \le \min\left\{\sqrt{\frac{m^m}{k_1^{k_1}\dots k_n^{k_n}}}\,,\frac{k_1^{k_1}\dots k_n^{k_n}}{m!}n^m\right\}
$$
for all non-negative integers $k_1,\dots,k_n$ with $k_1+\dots+k_n=m$.

Proof. The right hand side inequality is an immediate consequence of Proposition 2.3.
For the left hand side, let $x^i=(x_n^i)_{n=1}^\infty \in X$, $i=1,\dots,m$ and $l\in\mathcal{L}^s(^mX)$ be defined by
$$
L(x^1,\dots,x^m)=\frac{1}{m!}\sum_{\sigma\in S_m} x^1_{\sigma(1)}\dots x^m_{\sigma(m)},
$$
where S_m is the set of permutations of the first m natural numbers. Then
$$
L\big((x^1)^{k_1}\dots(x^n)^{k_n}\big)=\frac{1}{m!}\sum_{\sigma\in S_m} x^1_{\sigma(1)}\dots x^1_{\sigma(k_1)}\dots x^n_{\sigma(k_1+\dots k_{n-1}+1)}\dots x^n_{\sigma(k_1+\dots k_n)}.
$$
Take e_i to be the ith coordinate vector of X and define
$$
\begin{aligned}
y^1 &= \frac{1}{k_1}(e^1+\dots+e^{k_1})\\
y^2 &= \frac{1}{k_2}(e^{k_1+1}+\dots+e^{k_1+k_2})\\
&\vdots\\
y^n &= \frac{1}{k_n}(e^{k_1+\dots+k_{n-1}+1}+\dots+e^{k_1+\dots+k_n}).
\end{aligned}
$$
Then an easy calculation shows that $y^1,\dots,y^n$ are unit vectors in X and
$$
L\big((y^1)^{k_1}\dots(y^n)^{k_n}\big)=\frac{1}{m!}\frac{k_1!\dots k_n!}{k_1^{k_1}\dots k_n^{k_n}}.
$$

On the other hand, $\|\widehat{L}\| \leq \frac{1}{m^m}$, since

$$|\widehat{L}(x)| = |x_1 \ldots x_m| = \left[(|x_1| \ldots |x_m|)^{\frac{1}{m}}\right]^m \leq \left(\frac{|x_1| + \ldots + |x_m|}{m}\right)^m$$

by the arithmetic-geometric mean inequality. Thus

$$\left|L\big((y^1)^{k_1} \ldots (y^n)^{k_n}\big)\right| \geq \frac{k_1! \ldots k_n!}{k_1^{k_1} \ldots k_n^{k_n}} \frac{m^m}{m!} \|\widehat{L}\|.$$

$\square$

Remark 2.5. For $k_1 = \ldots = k_n = 1$, the upper and lower bound of $c(k_1, \ldots, k_n, X)$ in Theorem 2.4 give the same estimate which is $\frac{m^m}{m!}$.

3. POLYNOMIALS ON A REAL ℓ_p SPACE

In [9] Theorem 2, Y. Sarantopoulos proved that in the case of $L_p(\mu)$, for $1 \leq p \leq m'$, $\frac{1}{m} + \frac{1}{m'} = 1$ holds that

$$\|L\| \leq \frac{m^{\frac{m}{p}}}{m!} \|\widehat{L}\|.$$

This is an improved estimate of the one in [4] by L. A. Harris. Here we shall give some estimates of the constant $c(k_1, \ldots, k_n, \ell_p)$ for which $|L(x_1^{k_1} \ldots x_n^{k_n})| \leq c(k_1, \ldots, k_n, \ell_p) \|\widehat{L}\|$.

If f is a measurable function on $(X, \mathcal{A}, \mu)$, we define its distribution function $\lambda_f : (0, +\infty) \to [0, +\infty]$ by

$$\lambda_f(a) = \mu(\{x : |f(x)| > a\}).$$

From [3], we have the following

Proposition 3.1. If $\lambda_f(a) < \infty$ for every $a > 0$ and ϕ is a nonnegative Borel function on $(0, \infty)$, then

$$\int_X \phi \circ |f| d\mu = -\int_0^\infty \phi(a) d\lambda_f(a).$$

The case of this result we are interested in, is $\phi(a) = a^p$, which gives

$$\int |f|^p d\mu = -\int_0^\infty a^p d\lambda_f(a).$$

Integrating the right side by parts, we obtain

$$\int |f|^p d\mu = p \int_0^\infty a^{p-1} \lambda_f(a) da.$$

The validity of this calculation becomes clear if we consider that $a^p \lambda_f(a) \to 0$ as $a \to 0$ and $a \to \infty$ (since λ_f is strictly decreasing). In the following Proposition the function f will be of the form $f(t) = r_1(t) + \ldots + r_k(t)$, $k \in \mathbb{N}$. Therefore, using Hoeffding's inequality (see [6] Theorem 2), we get that

$$\lambda_f(x) := \lambda_k(x) = P(|r_1(t) + \ldots + r_k(t)| \geq x) \leq 2e^{-\frac{x^2}{2k}}.$$

Proposition 3.2. Let $1 \leq p \leq \infty$ and $L : (\ell_p)^m \to \mathbb{R}$ be a continuous symmetric m-linear mapping with associated homogeneous polynomial $\widehat{L}$. If $x_1, \ldots, x_n$ are norm-one vectors in ℓ_p with disjoint supports, then

$$\frac{|L(x_1^{k_1} \ldots x_n^{k_n})|}{\|\widehat{L}\|} \leq \begin{cases} \min\left\{\frac{k_1^{k_1} \ldots k_n^{k_n}}{m!} n^{\frac{m}{p}}, \frac{p 2^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right) m^{\frac{p}{2}}}{m!}\right\}, & \text{if } p \geq m \\ \min\left\{\frac{k_1^{k_1} \ldots k_n^{k_n}}{m!} n^{\frac{m}{p}}, \frac{n^{\frac{m-p}{p}} m 2^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right) m^{\frac{m}{2}}}{m!}\right\}, & \text{if } p < m \end{cases}$$

for all non-negative integers $k_1, \ldots, k_n$ with $k_1 + \ldots + k_n = m$.

Proof. Working as in the proof of Proposition 2.3, we get

(5)
$$\begin{aligned} |L(x_1^{k_1} \ldots x_n^{k_n})| &\leq \frac{k_1^{k_1} \ldots k_n^{k_n}}{m!} \|\widehat{L}\| \int_0^1 \left\| (r_1(t) + \ldots + r_{k_1}(t)) \frac{x_1}{k_1} + \ldots \right\|_p^m dt \\ &\leq \frac{k_1^{k_1} \ldots k_n^{k_n}}{m!} \|\widehat{L}\| \int_0^1 \left[\left(\frac{|r_1(t) + \ldots + r_{k_1}(t)|}{k_1} \right)^p + \ldots \right]^{\frac{m}{p}} dt \\ &\leq \frac{k_1^{k_1} \ldots k_n^{k_n}}{m!} n^{\frac{m}{p}} \|\widehat{L}\|. \end{aligned}$$

We shall now use a different technique and we need to distinguish two cases.

For $p \geq m$:

$$\begin{aligned} |L(x_1^{k_1} \ldots x_n^{k_n})| &\leq \frac{\|\widehat{L}\|}{m!} \int_0^1 \|(r_1(t) + \ldots + r_{k_1}(t)) x_1 + \ldots\|_p^m dt \\ &\leq \frac{\|\widehat{L}\|}{m!} \int_0^1 (|r_1(t) + \ldots + r_{k_1}(t)|^p + \ldots)^{\frac{m}{p}} dt \\ &\leq \frac{\|\widehat{L}\|}{m!} \left(p \int_0^\infty x^{p-1} \lambda_{k_1}(x) dx + \ldots + p \int_0^\infty x^{p-1} \lambda_{k_n}(x) dx \right) \\ &\leq \frac{\|\widehat{L}\|}{m!} \left(p \int_0^\infty x^{p-1} 2 e^{-\frac{x^2}{2k_1}} dx + \ldots + p \int_0^\infty x^{p-1} 2 e^{-\frac{x^2}{2k_n}} dx \right) \\ &= \frac{\|\widehat{L}\|}{m!} \left(p (2k_1)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right) + \ldots + p (2k_n)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right) \right) \\ &= \frac{p 2^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right)}{m!} \sum_{i=1}^n k_i^{\frac{p}{2}} \|\widehat{L}\| \leq \frac{p 2^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right)}{m!} \left(\sum_{i=1}^n k_i \right)^{\frac{p}{2}} \|\widehat{L}\| \\ &= \frac{p 2^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right) m^{\frac{p}{2}}}{m!} \|\widehat{L}\| \end{aligned}$$

For $p < m$:

$$\begin{aligned}
|L(x_1^{k_1}\dots x_n^{k_n})| &\le \frac{\|\widehat{L}\|}{m!}\int_0^1 \|(r_1(t)+\dots+r_{k_1}(t))x_1+\dots\|_p^m dt \\
&\le \frac{\|\widehat{L}\|}{m!}\int_0^1 (|r_1(t)+\dots+r_{k_1}(t)|^p+\dots)^{\frac{m}{p}} dt \\
&= \frac{n^{\frac{m}{p}}\|\widehat{L}\|}{m!}\int_0^1 \left(\frac{|r_1(t)+\dots+r_{k_1}(t)|^p+\dots}{n}\right)^{\frac{m}{p}} dt \\
&\le \frac{n^{\frac{m}{p}}\|\widehat{L}\|}{m!}\int_0^1 \frac{|r_1(t)+\dots+r_{k_1}(t)|^m+\dots}{n} dt \\
&\le \frac{n^{\frac{m-p}{p}}\|\widehat{L}\|}{m!}\left(m\int_0^\infty x^{m-1}\lambda_{k_1}(x)dx+\dots+m\int_0^\infty x^{m-1}\lambda_{k_n}(x)dx\right) \\
&\le \frac{n^{\frac{m-p}{p}}\|\widehat{L}\|}{m!}\left(m\int_0^\infty x^{m-1}2e^{-\frac{x^2}{2k_1}}dx+\dots+m\int_0^\infty x^{m-1}2e^{-\frac{x^2}{2k_n}}dx\right) \\
&= \frac{n^{\frac{m-p}{p}}\|\widehat{L}\|}{m!}\left(m(2k_1)^{\frac{m}{2}}\Gamma\big(\frac{m}{2}\big)+\dots+m(2k_n)^{\frac{m}{2}}\Gamma\big(\frac{m}{2}\big)\right) \\
&= \frac{n^{\frac{m-p}{p}}m2^{\frac{m}{2}}\Gamma\big(\frac{m}{2}\big)}{m!}\sum_{i=1}^n k_i^{\frac{m}{2}}\|\widehat{L}\| \le \frac{n^{\frac{m-p}{p}}m2^{\frac{m}{2}}\Gamma\big(\frac{m}{2}\big)}{m!}\left(\sum_{i=1}^n k_i\right)^{\frac{m}{2}}\|\widehat{L}\| \\
&= \frac{n^{\frac{m-p}{p}}m2^{\frac{m}{2}}\Gamma\big(\frac{m}{2}\big)m^{\frac{m}{2}}}{m!}\|\widehat{L}\|
\end{aligned}$$

□

Note that the second technique used in the proof of Proposition 3.2 gives better estimates than (5) for large m's and small n's and its special case where $p \ge m$ holds asymptotically only for ℓ_∞. Moreover, observe that Proposition 3.2 gives better estimates than Proposition 2.3.

Remark 3.3. The term $\sum_{i=1}^n k_i^{\frac{p}{2}}$ (resp. $\sum_{i=1}^n k_i^{\frac{m}{2}}$) in the proof of Proposition 3.2 can be bounded by $(m-n+1)^{\frac{p}{2}}+n-1$ (resp. $(m-n+1)^{\frac{m}{2}}+n-1$) instead of $m^{\frac{p}{2}}$ (resp. $m^{\frac{m}{2}}$) since it takes its maximum value when all but one k_i's equal to 1 and the last one equals to $m-n+1$.

We now conclude that:

Theorem 3.4. Let $1 \le p \le \infty$ and $L : (\ell_p)^m \to \mathbb{R}$ be a continuous symmetric m-linear mapping with associated homogeneous polynomial $\widehat{L}$. If $x_1, \dots, x_n$ are norm-one vectors in ℓ_p with disjoint supports, then:
if $p \ge m$

$$\frac{k_1!\dots k_n!}{k_1^{\frac{k_1}{p}}\dots k_n^{\frac{k_n}{p}}}\frac{m^{\frac{m}{p}}}{m!} \le c(k_1,\dots,k_n,\ell_p) \le \min\left\{\frac{k_1^{k_1}\dots k_n^{k_n}}{m!}n^{\frac{m}{p}}, \frac{p2^{\frac{p}{2}}\Gamma\big(\frac{p}{2}\big)m^{\frac{p}{2}}}{m!}\right\}$$

and if $p < m$

$$\frac{k_1! \dots k_n!}{k_1^{\frac{k_1}{p}} \dots k_n^{\frac{k_n}{p}}} \frac{m^{\frac{m}{p}}}{m!} \leq c(k_1, \dots, k_n, \ell_p) \leq \min\left\{ \frac{k_1^{k_1} \dots k_n^{k_n}}{m!} n^{\frac{m}{p}}, \frac{n^{\frac{m-p}{p}} m 2^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right) m^{\frac{m}{2}}}{m!} \right\}$$

for all non-negative integers $k_1, \dots, k_n$ with $k_1 + \dots + k_n = m$.

Proof. The right hand side inequality is an immediate consequence of Proposition 3.2.
For the left hand side, we just need some simple adjustments to the proof of Proposition 2.3. □

Remark 3.5. For $k_1 = \dots = k_n = 1$, the upper and lower bound of $c(k_1, \dots, k_n, \ell_p)$ in Theorem 3.4 give the same estimate which is $\frac{m^{\frac{m}{p}}}{m!}$.

4. Radius of analyticity of a power series on a real Banach space

A power series centered at $a \in X$ is a formal sum

$$\sum_{m=0}^{\infty} P_m(x-a) \tag{6}$$

where for each m, $P_m : X \to Y$ is a continuous m-homogeneous polynomial.

The radius of uniform convergence of this power series is defined to be

$$\rho := \{r : (6) \text{ converges uniformly on } |x-a| \leq r\},$$

which is given by the following standard formula:

$$\rho = \frac{1}{\limsup_{m\to\infty} \|P_m\|^{\frac{1}{m}}}.$$

If $\rho > 0$, then for every $0 < r < \rho$, (6) is a uniformly and absolutely convergent series for every $x \in B_r(a)$. hence in this case, (6) defines a function on $B_\rho(a)$ taking values in Y.

The Taylor series of an infinitely differentiable function F defined in a neighborhood of a is the power series defined by

$$T_aF(x) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m F(a)((x-a)^m),$$

where $D^m F(a) : X^m \to Y$ is the symmetric m-linear map given by taking the Fréchet derivative of F m times.

F is called analytic at a if $T_aF(x)$ has a positive radius of uniform convergence and equals $F(x)$ within the domain of uniform convergence. If $U \subset X$ is open, we say that F is analytic in U if it is analytic at every $a \in U$. If furthermore, $T_aF(x)$ converges uniformly in every closed ball centered at a contained in U, for each $a \in U$, F is called fully analytic in U.

Let $F(x)$ be a power series centered at a with radius of uniform convergence $\rho > 0$. The radius of analyticity $\rho_A = \rho_A(F)$ of $F(x)$ at a is the largest $r > 0$ such that $F(x)$ is fully analytic in $B_r(a)$.

The norm we need to control when we expand a power series at a new point is $\|L\|_{(2)}$. To see this, consider the power series (6) centered at $a = 0$, which we may rewrite as

$$F(x) = \sum_{m=0}^{\infty} L_m(x^m). \tag{7}$$

Observe that by the binomial formula, given any $y \in X$

$$L_m(x^m) = L_m((y + x - y)^m) = \sum_{k=0}^{\infty} \binom{m}{k} L_m(y^{m-k}, (x-y)^k).$$

Then, if in

$$\sum_{m=0}^{\infty} L_m(x^m) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} L_m(y^{m-k}, (x-y)^k) \tag{8}$$

the double series on the right converges absolutely, we can interchange summations and obtain

$$F(x) = \sum_{m=0}^{\infty} L_m(x^m) = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \binom{m}{k} L_m(y^{m-k}, (x-y)^k)$$

Thus, if we can perform this change of summation for all $x \in B_r(y)$, for some $r > 0$, then we will have expressed $F(x)$ as a power series centered at y whose k-homogeneous polynomial coefficients are given by

$$A_k(z) := \sum_{m=k}^{\infty} \binom{m}{k} L_m(y^{m-k}, z^k). \tag{9}$$

Observe that the absolute convergence of the double sum (8) for $x \in B_r(y)$ implies the absolute convergence of the A_k in $B_r(0)$ and hence on all X by homogeneity.

Absolute convergence of (8) holds if

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \|L_m\|_{(2)} |y|^{m-k} |x-y|^k = \sum_{m=0}^{\infty} \|L_m\|_{(2)} (|y| + |x-y|)^m < \infty.$$

This holds when

$$|y| + |x - y| < \frac{1}{\limsup \|L_m\|_{(2)}^{\frac{1}{m}}}. \tag{10}$$

Choose a subsequence m_i such that

$$\lim_{i \to \infty} \|L_{m_i}\|_{(2)}^{\frac{1}{m_i}} = \limsup \|L_m\|_{(2)}^{\frac{1}{m}}.$$

Let $\rho = \frac{1}{\limsup \|\widehat{L}_m\|_{(2)}^{\frac{1}{m}}} > 0$ be the radius of uniform convergence of (7) and suppose $\rho < \infty$. Then (10) is satisfied if

$$\begin{aligned} |y| + |x-y| &< \rho \cdot \frac{\limsup_{i\to\infty} \|\widehat{L}_{m_i}\|^{\frac{1}{m_i}}}{\lim_{i\to\infty} \|L_{m_i}\|_{(2)}^{\frac{1}{m_i}}} \\ &= \rho \cdot \limsup_{i\to\infty} \left(\frac{\|\widehat{L}_{m_i}\|}{\|L_{m_i}\|_{(2)}} \right)^{\frac{1}{m_i}} \\ &:= \bar{\rho}. \end{aligned} \tag{11}$$

Thus, for $|y| < \bar{\rho}$ and $|x-y| < \bar{\rho} - |y|$, the series (8) converges absolutely.

Altogether then, we have shown that for any fixed $|y| < \bar{\rho}$, we have

$$F(x) = \sum_{k=0}^{\infty} A_k(x-y) \tag{12}$$

for $|x-y| < \bar{\rho} - |y|$.

From [1] Corollary 1, p. 165, we have the following

Lemma 4.1. For any power series $F(x) = \sum_{k=0}^{\infty} A_k(x-y)$ centered at y with positive radius of uniform convergence, $A_k = \frac{1}{k!} D^k F(y)$ as k-homogeneous polynomials.

Theorem 4.2. Let $F(x)$ be a power series in a Banach space X, which we may take to be centered at the origin. Let $\rho > 0$ denote its radius of uniform convergence and ρ_A its radius of analyticity. Then

(i) $\rho_A \geq \frac{\rho}{\sqrt{2}}$,

(ii) for every n, the nth Fréchet derivative $D^n F : X \to \mathcal{L}_n(X, Y)$ of $F(x)$, viewed as a map from X to the Banach space $\mathcal{L}_n(X, Y)$ of continuous n-linear maps from X to Y, has a Taylor series centered at the origin with radius of uniform convergence at least $\frac{\rho}{\sqrt{2}}$ for $n = 2$ and $\frac{\rho}{\sqrt{e}}$ for $n \geq 3$,

(iii) the radius of analyticity of the power series $D^n F(x)$ is at least $\frac{\rho}{\sqrt{2}}$.

Proof. (i) Suppose $\rho < \infty$. Lemma 2.1 implies

$$\limsup_{m\to\infty} \left(\frac{\|\widehat{L}_m\|}{\|L_m\|_{(2)}} \right)^{\frac{1}{m}} \geq \frac{1}{\sqrt{2}}, \tag{13}$$

hence $\bar{\rho} \geq \frac{\rho}{\sqrt{2}}$ by (11). Thus the preceding analysis shows that for $|y| < \rho$, the Taylor series (12) is absolutely convergent in $\{x : |x-y| < \frac{\rho}{\sqrt{2}} - |y|\}$. From this, we get uniform convergence of (12) in $\{x : |x-y| < r\}$ for every $r < \frac{\rho}{\sqrt{2}} - |y|$, since one can bound the

tail of (12) as follows:

$$\left|\sum_{k=N}^{\infty} A_k(x-y)\right| \leq \sum_{k=N}^{\infty}\sum_{m=k}^{\infty}\binom{m}{k}|L_m(y^{m-k},(x-y)^k)| \\ \leq \sum_{m=N}^{\infty}\|L_m\|_{(2)}(|y|+|x-y|)^m \tag{14}$$

where (14) tends to zero uniformly in x as $N \to 0$ so long as $|y|+|x-y| \leq |y|+r$ is bounded away from $\bar{\rho} \geq \frac{\rho}{\sqrt{2}}$. This follows because we have shown that (14) viewed as power series in a single real variable has radius of uniform convergence at least $\bar{\rho}$. Finally, Lemma 4.1 implies that the power series (12) is a Taylor series of $F(x)$ centered at y. So for $\rho < \infty$, this proves (i) since we have shown $\rho_A \geq \frac{\rho}{\sqrt{2}}$.

For the remaining case $\rho = \infty$, i.e. $\limsup_{m\to\infty}\|\widehat{L}_m\|^{\frac{1}{m}} = 0$, (13) implies also that $\limsup_{m\to\infty}\|L_m\|_{(2)}^{\frac{1}{m}} = 0$. From (10), we can apply the previous analysis for every $\bar{\rho} > 0$, hence $\rho_A = \rho = \infty$.

(ii) By Chae, given any power series $F(x) = \sum_{m=0}^{\infty} L_m(x^m)$ with radius of uniform convergence $\rho > 0$, for every n, $D^nF(x)$ has a Taylor series centered at the origin given by

$$T_0D^nF(x) = n!\sum_{m=0}^{\infty}\binom{m+n}{n}L_{m+n}(x^m). \tag{15}$$

In (15), the linear maps L_{m+n} are only evaluated on at most $n+1$ distinct arguments ($D^nF(x)$ takes values in n-linear maps). So by Lemma 2.1, for $n \geq 3$, the lower bound for the radius of uniform convergence of (15) is $\frac{\rho}{\sqrt{e}}$ since

$$\begin{aligned}\limsup_{m\to\infty}\left\|\binom{m+n}{n}L_{m+n}\right\|_{(n+1)}^{\frac{1}{m}} &\leq \left(\limsup_{m\to\infty} C(m,n+1)\sqrt{e}\right) \\ &\quad \times\left(\limsup_{m\to\infty}\|\widehat{L}_{m+n}\|^{\frac{1}{m}}\right) \\ &\leq \frac{\sqrt{e}}{\rho}.\end{aligned}$$

Similarly, the lower bound for the radius of uniform convergence of $T_0DF(x) = \sum_{m=0}^{\infty}(m+1)L_{m+1}(x^m)$ is $\frac{\rho}{\sqrt{2}}$ since $\limsup_{m\to\infty}\|(m+1)L_{m+1}\|_{(2)}^{\frac{1}{m}} \leq \frac{\sqrt{2}}{\rho}$.

By Lemma 4.1 and the formula (9) for the A_k, it follows that for $n \geq 3$, $T_0D^nF(y) = D^nF(y)$ for all y such that $|y| < \frac{\rho}{\sqrt{e}}$ and $T_0DF(y) = DF(y)$ for all y such that $|y| < \frac{\rho}{\sqrt{2}}$.

(iii) We need to show that the radius of analyticity of (15) is at least $\frac{\rho}{\sqrt{2}}$. For this, we need to control $\|L_{m+n}\|_{(n+2)}$, but this is precisely

the $\|\cdot\|_{(2)}$-norm of L_{m+n} viewed as a map from X to $\mathcal{L}_n(X,Y)$. Applying Lemma 2.1 the proof is completed.

□

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